Indian Statistical Institute, Bangalore Centre
B.Math(Hons.) I Year, First Semester
Mid-Sem Examination
Analysis I

Time: 3 Hours | September 23, 2011 | Instructor: C.R.E.Raja
Total Marks: 40

Answer any five questions, each question is worth 8 marks :

1. Let (a_n) and (b_n) be sequences converging to a and b respectively. Prove that
   (i) $a_n + b_n \to a + b$ and $ra_n \to ra$ for any $r \in \mathbb{R}$,
   (ii) (a_n) is a bounded sequence.

2. Let (x_n) and (y_n) be bounded sequences. Prove that $\underline{\lim}(-x_n) = -\overline{\lim}x_n$ and
$$\begin{aligned}\underline{\lim}x_n + \underline{\lim}y_n \leq \underline{\lim}(x_n + y_n) &\leq \overline{\lim}x_n + \underline{\lim}y_n \\ &\leq \overline{\lim}(x_n + y_n) \leq \overline{\lim}x_n + \overline{\lim}y_n.\end{aligned}$$

3. (i) If (a_n) is a sequence and r is a limit point of (a_n), then show that there is a subsequence (a_{k_n}) of (a_n) such that $a_{k_n} \to r$.
   (ii) If (a_n) is defined by
$$a_1 = 0, \quad a_{2m} = \frac{a_{2m-1}}{2}, \quad a_{2m+1} = \frac{1}{2} + a_{2m}$$
   for any $m \geq 1$. Find $\liminf a_n$ and $\limsup a_n$.

4. (i) Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty}(n-1)^{\frac{1}{n}} = 1$.
   (ii) Prove that every Cauchy sequence converges.

5. (i) If $|a_n| \leq c_n$ for all n and $\sum c_n$ converges, prove that $\sum a_n$ also converges.
   (ii) Let (a_i) be a decreasing sequence of non-negative numbers. Then prove that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^k a_{2^k}$ converges.

6. (i) If the sequence of partial sums of $\sum a_n$ is bounded and (b_n) is a decreasing or increasing sequence converging to zero, prove that $\sum a_n b_n$ converges.
   (ii) If $\sum a_n$ converges and (b_n) is a bounded monotonic sequence, prove that $\sum a_n b_n$ converges.

7. (i) Find the radius of convergence of the following series

$$\frac{1}{3} + \frac{1}{5}z + \frac{1}{3^2}z^2 + \frac{1}{5^2}z^3 + \frac{1}{3^3}z^4 + \frac{1}{5^3}z^5 + \cdots$$

(ii) Let (a_n) be a sequence. Define $p_n = |a_n| + a_n$ and $q_n = |a_n| - a_n$. Prove that

(a) $\sum p_n$ and $\sum q_n$ converge if and only if $\sum a_n$ converges absolutely;

(b) if $\sum a_n$ and $\sum p_n$ converge, then $\sum a_n$ converges absolutely.